RAYLEIGH WAVES IN PRESTRESSED MEDIUM

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ABSTRACT

The propagation of Rayleigh waves in cylindrical co-ordinate in a prestressed medium due to initial compressive stresses is discussed using Hankel integral transformation. The effect of initial stresses on the displacements is studied deeply.

***KEYWORDS:-** Rayleigh waves, Prestressed medium, Hankel Transform , displacement components .*

INTRODUCTION

After pioneering work of Rayleigh , Knowles [1] solved the Rayleigh problem in terms of single potential and this single formulation has been extended by Eringen and Suhubi [2].The theory of Rayleigh waves has been widely summarized in the book of Ewing et al.[4] ,Achenbach [5], Pilant [3]and Ben-Menahem and singh [6].

Biot [7] investigated the effect of gravity on Rayleigh waves .He assumes that gravity creates a initial stresses of hydrostatic nature. Adapting the same theory of initial stress and using the dynamical equations of motion for the initial compressive stress, Chattopadhyay et al.[8] discussed the propagation of Rayleigh waves in cylindrical coordinates. The authors solved this problem by introducing potential method, which is, in general , not applicable in prestressed media.

The present paper is a sincere effort the propagation of Rayleigh waves in cylindrical coordinates in a prestressed medium with help of Hankel transformation .Here it also drives the expressions for frequency equation and displacement components of Rayleigh waves respectively.

FORMULATION AND PROBLEM OF THE SOLUTION

For an axi–symmetry problem $v = 0$, and $u \; and \; w$ are independent of θ the equations of motion under initial compressive stress $\sigma_{33} = -P$ along the z-direction only in cylindrical polar coordinates are given by (Biot,1965).

$$\frac{\sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - P\frac{\partial \overline{\omega\theta}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\sigma_{rz}}{\partial r}+\frac{\partial \sigma_{rz}}{r}+\frac{\partial \sigma_{zz}}{\sigma z}-P\left(\frac{\partial \overline{\omega\theta}}{\partial r}+\frac{\overline{\omega\theta}}{r}\right)=\rho\frac{\partial^2 \omega}{\partial t^2}\,, \tag{1}$$

where σ_{ij} are the incremental stress components along rotated direction of the axes and

$$\omega_\theta=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial \omega}{\partial r}\right),$$

$$\sigma_{rr}=(\lambda+2\mu)\frac{\partial u}{\partial r}+\lambda\left(\frac{u}{r}+\frac{\partial \omega}{\partial z}\right),$$

$$\sigma_{\theta\theta}=(\lambda+2\mu)\frac{u}{r}+\lambda\left(\frac{\partial u}{\partial r}+\frac{\partial \omega}{\partial z}\right), \tag{2}$$

$$\sigma_{zz}=(\lambda+P)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+(\lambda+2\mu+P)\frac{\partial \omega}{\partial z},$$

$$\sigma_{rz}=\mu\left(\frac{\partial \omega}{\partial r}+\frac{\partial u}{\partial z}\right),$$

where $u(r,z,t)$ *and* $\omega(r,z,t)$ *are* the displacements in the incremental stage along the r *and* z directions and ρ is the density Using equation (2) in equation (1) , we get

$$(\lambda+2\mu)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 \omega}{\partial r\partial z}\right]+\left(\mu-\frac{P}{2}\right)\left[\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 \omega}{\partial r\partial z}\right]=\rho\frac{\partial^2 u}{\partial t^2},$$

(3)

$$\left(\mu+\frac{P}{2}\right)\left[\frac{\partial^2 \omega}{\partial r^2}+\frac{1}{r}\frac{\partial \omega}{\partial r}-\frac{\partial^2 u}{\partial r\partial z}-\frac{1}{r}\frac{\partial u}{\partial z}\right]+(\lambda+2\mu+P)\left[\frac{\partial^2 \omega}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial r\partial z}\right]=\rho\frac{\partial^2 \omega}{\partial t^2}\,.$$

We assume the motion is harmonic and so we can write

$$u=u(r,z,t)=U(r,z)e^{-i\omega t},$$

(4)

$$w=w(r,z,t)=W(r,z)e^{-i\omega t}\,,$$

where ω is a frequency parameter. Putting the values of u *and* w from equation (4) in equation (3) we get

$$(\lambda+2\mu)\left[\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{U}{r^2}+\frac{\partial^2 W}{\partial r\partial z}\right]+\left(\mu-\frac{P}{2}\right)\left[\frac{\partial^2 U}{\partial z^2}-\frac{\partial^2 W}{\partial r\partial z}\right]=-\rho\omega^2 U,$$

(5)

$$\left(\mu+\frac{P}{2}\right)\left[\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial W}{\partial r}-\frac{\partial^2 U}{\partial r\partial z}-\frac{\partial U}{\partial z}\right]+(\lambda+2\mu+P)\left[\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial z}+\frac{\partial^2 U}{\partial r\partial z}\right]=-\rho\omega^2 W\,.$$

We define the Hankel Transformation as

$$U_H(k,z) = \int_0^\infty rU(r,z)\, J_1(kr)\, dr,$$

$$W_H(k,z) = \int_0^\infty rW(r,z)\, J_0(kr)\, dr, \tag{6}$$

where $U_H(k,z)$ and $W_H(k,z)$ are displacement transforms. Using the Hankel integral Transformation from equation (6) in first equation of (5), we get

$$\left(\rho\omega^2 - k^2(\lambda+2\mu) + \left(\mu + \frac{P}{2}\right)\frac{d^2}{dz^2}\right) U_H + \left(\lambda + \mu + \frac{P}{2}\right)(-k)\,\frac{dW_H}{dz} = 0\,. \tag{7}$$

We assume

$$U_H = \rho k(\alpha^2 - \beta_1^2)\frac{dF}{dz},$$

$$W_H = \rho\,(\omega^2 - \alpha^2 k^2 + \beta_1^2 \frac{d^2}{dz^2})F, \tag{8}$$

where F is an auxiliary function and assumptions lead to differential equation

$$\frac{d^4F}{dz^4} - \left(\left(2k^2 - \omega^2\left(\frac{1}{\alpha^2} + \frac{1}{\beta_1^2}\right)\right)\frac{d^2F}{dz^2} + k^4 - \omega^2 k^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta_1^2}\right) + \frac{\omega^4}{\alpha^2\beta_1^2}\right) = 0. \tag{9}$$

where

$$\alpha^2 = \frac{\lambda+2\mu}{\rho}, \quad \beta_1^2 = \beta^2(1-I),$$

$$\beta^2 = \frac{\mu}{\rho}, \tag{10}$$

$$I = \frac{P}{2\mu}\,.$$

Then F can be written as

$$F(z) = A_1 e^{qz} + A_2 e^{-qz} + A_3 e^{sz} + A_4 e^{-sz}, \tag{11}$$

where

$$q = \left(k^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2},$$

$$s = \left(k^2 - \frac{\omega^2}{\beta_1^2}\right)^{1/2}. \tag{12}$$

BOUNDARY CONDITIONS

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incremental boundary conditions are

$$\Delta f_z = \sigma_{zz} - P(e_{rr} + e_{\theta\theta}) \ ,$$

$$= \sigma[(\alpha^2 + 2\beta^2 I)\frac{\partial w}{\partial z} + (\alpha^2 - \beta^2 - 2\beta^2 I)(\frac{\partial u}{\partial r} + \frac{u}{r}),$$

$$= 0 \ , \qquad \text{at } z = 0 \qquad (13)$$

$$\Delta f_r = \mu\sigma_{zr} \ ,$$

$$= \mu(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}) \ ,$$

$$= 0,$$

where

Δf_z *and* Δf_r are incremental boundary forces per unit area . These can be written as

$$\Delta f_r = \sigma_{rr}\eta_r + \sigma_{r\theta}\eta_\theta + \sigma_{rz}\eta_z + \overline{\sigma_{33}}\ \overline{\omega_\theta}\ \eta_r \ ,$$

$$(14)$$

$$\Delta f_z = \sigma_{zr}\eta_r + \sigma_{z\theta}\eta_\theta + \sigma_{zz}\eta_z + \overline{\sigma_{33}}\,(e_{rr} + e_{\theta\theta})\eta_r - \overline{\sigma_{33}}\,(e_{zr})\eta_r - \overline{\sigma_{33}}\,(e_{\theta\theta})\eta_\theta,$$

where η_r, η_θ *and* η_z are cosines of the angle between normal to the deformation boundary in the r, θ *and* z directions before the incremental deformation. $e_{rr}, e_{\theta\theta}$ *and* e_{zr} defined as

$$e_{rr} = \frac{\partial u}{\partial z}, e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \quad and \ e_{zr} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right). \qquad (15)$$

For Rayleigh waves, we consider compressional and distortional waves along the r axis only and call them P and SV waves, respectively then equation (2) from equation (9) can be written as

$$F(z) = A_2 e^{-qz} + A_4 e^{-sz} \quad , \qquad (16)$$

using the value of $F(z)$ form equation(16) in equation(8) we get

$$U_H = -\rho k(\alpha^2 - \beta_1^2)[A_2 q e^{-qz} + A_4 s e^{-sz}\ ] \quad , \qquad (17)$$

$$W_H = \rho[A_2(\omega^2 - \alpha^2 k^2 + \beta_1^2 q^2)e^{-qz} + A_4(\omega^2 - \alpha^2 k^2 + \beta_1^2 s^2)e^{-sz}] \ ,$$

After obtaining the Hankel integral transform of equation (13) and simplifying, we get

$(-k)W_H + \frac{\partial U_H}{\partial z} = 0$,

at z= 0

$$-k(\alpha^2 - \beta_1^2 - 3\beta^2 I)U_H + (\alpha^2 + 2\beta^2 I)\frac{dW_H}{dz} = 0 \quad . \tag{18}$$

Putting the value of equation (17) in equation (18), we get

$$A_2[(\omega^2 - \alpha^2(k^2+q^2) + \beta_1^2 q^2)] + A_4[(\omega^2 - \alpha^2(k^2+s^2) + \beta_1^2 s^2)] = 0,$$

$$[k^2(\alpha^2 - \beta^2 + \beta^2 I)(\alpha^2 - \beta^2 - 2\beta^2 I) - (\alpha^2 + 2\beta^2 I)\{(\omega^2 - \alpha^2 k^2) + \beta^2 q^2(1 - I)\}q]A_2 +$$

$$[k^2(\alpha^2 - \beta^2 + \beta^2 I)(\alpha^2 - \beta^2 - 2\beta^2 I) - (\alpha^2 + 2\beta^2 I)\{(\omega^2 - \alpha^2 k^2) + \beta^2 s^2(1 - I)\}s]A_4 = 0 \, . \tag{19}$$

Eliminating A_2 and A_4 from equation (19), we obtain the frequency equation of Rayleigh waves

$$\left[\frac{c^2}{\beta^2} - 2\nu + (2 - 2I - \nu)\left(1 - \frac{c^2}{\beta^2 \nu}\right)\right]\left[(\nu - 1 - I)(\nu - 1 - 2I) - (\nu + 2I)\left\{\left(\frac{c^2}{\beta^2} - \nu\right) + \left(1 - \frac{c^2}{\beta^2(1-I)}\right)(1 - I)\right\}\sqrt{\left(1 - \frac{c^2}{\beta^2(1-I)}\right)}\right] - \left[\frac{c^2}{\beta^2} - 2\nu + (2 - 2I - \nu)\left(1 - \frac{c^2}{\beta^2(1-I)}\right)\right]\left[(\nu - 1 - I)(\nu - 1 - 2I) - (\nu + 2I)\left\{\left(\frac{c^2}{\beta^2} - \nu\right) + \left(1 - \frac{c^2}{\beta^2 \nu}\right)(1 - I)\right\}\sqrt{\left(1 - \frac{c^2}{\beta^2 \nu}\right)}\right] = 0 \, , \tag{20}$$

where = $\frac{\alpha^2}{\beta^2}$.

The displacement components are

$$u(r,z,t) = A_2\rho(\alpha^2 - \beta^2)[1 - I]e^{-i\omega t}\int_0^\infty k\left\{\sqrt{\left(1 - \frac{c^2}{\beta^2}\right)}\, e^{-k\sqrt{\left(1 - \frac{c^2}{\alpha^2}\right)}\, z} + M\sqrt{\left(1 - \frac{c^2}{\beta^2(1-I)}\right)}\, e^{-k\sqrt{\left(1 - \frac{c^2}{\beta^2(1-I)}\right)}\, z}\right\}J_1(kr)dk$$

,

$$w(r,z,t) = A_2\rho e^{-i\omega t}\int_0^\infty k\left[\left\{c^2 - \alpha^2 + \beta^2(1 - I)\left(1 - \frac{c^2}{\alpha^2}\right)\right\}e^{-k\sqrt{\left(1 - \frac{c^2}{\alpha^2}\right)}\, z} + M\left\{c^2 - \alpha^2 + \beta^2(1 - I)\left(1 - \frac{c^2}{\beta^2(1 - I)}\right)\right\}e^{-k\sqrt{\left(1 - \frac{c^2}{\beta^2(1-I)}\right)}\, z}\right]J_0(kr)dk \, , \tag{21}$$

where

$$M = \frac{c^2 - \alpha^2 + (2\beta^2(1 - I) - \alpha^2)\left(1 - \frac{c^2}{\alpha^2}\right)}{c^2 - \alpha^2 + (2\beta^2(1 - I) - \alpha^2)\left(1 - \frac{c^2}{\beta^2(1-I)}\right)} \, .$$

NUMERICAL CALCULATION AND DISCUSSION

For numerical calculations we take kr = 6.5 and the $\boldsymbol{\nu = 0.25}$. For different value of I e.g. 0.0,0.2,0.6 ,the $\frac{c}{\beta}$ can be calculated from equation(20) .It is clear that velocity ratio decreases as the initial stress increases . The relative displacement $\frac{u}{ka}$ and $\frac{w}{ka}$ are calculated numerically and represented graphically as below .

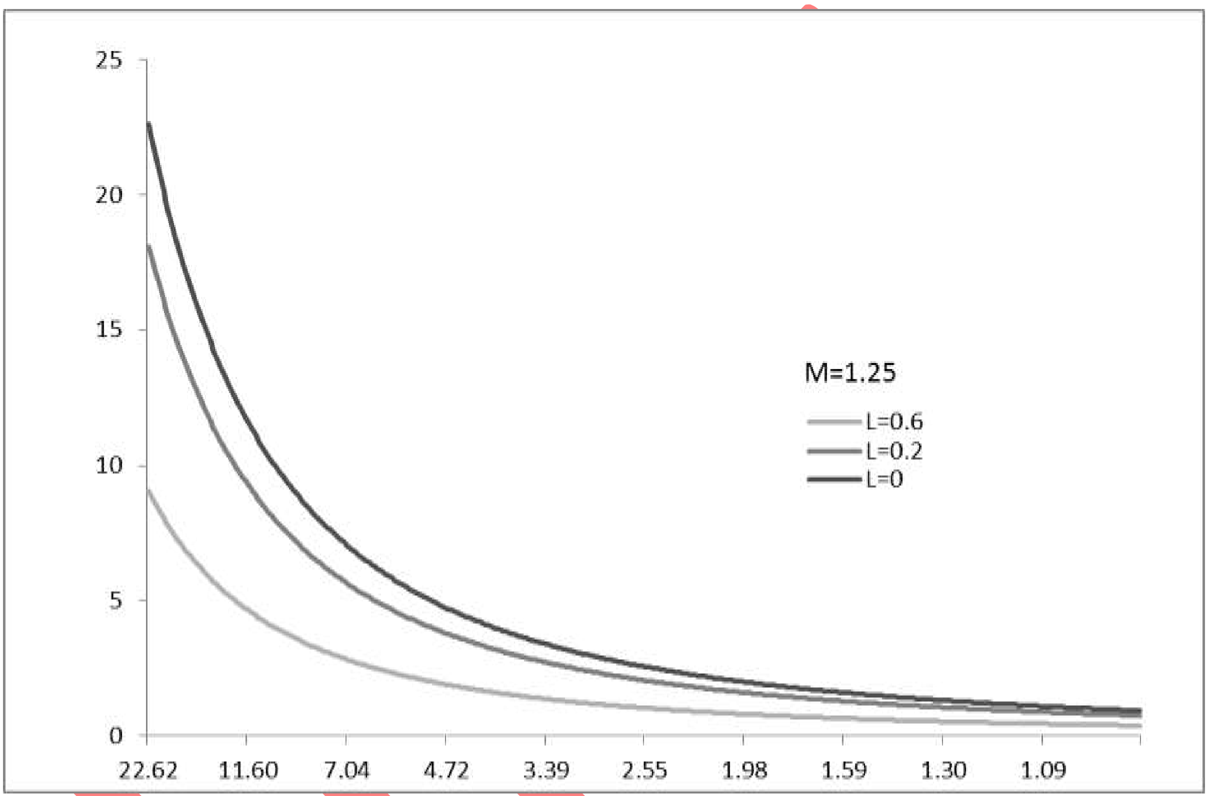

Fig.1 Variation of Displacement $\frac{u}{ka} = \boldsymbol{U}$ with z given kr = 6.5 , $\boldsymbol{\nu = 0.25}$ and I=0.0,0.2,0.6

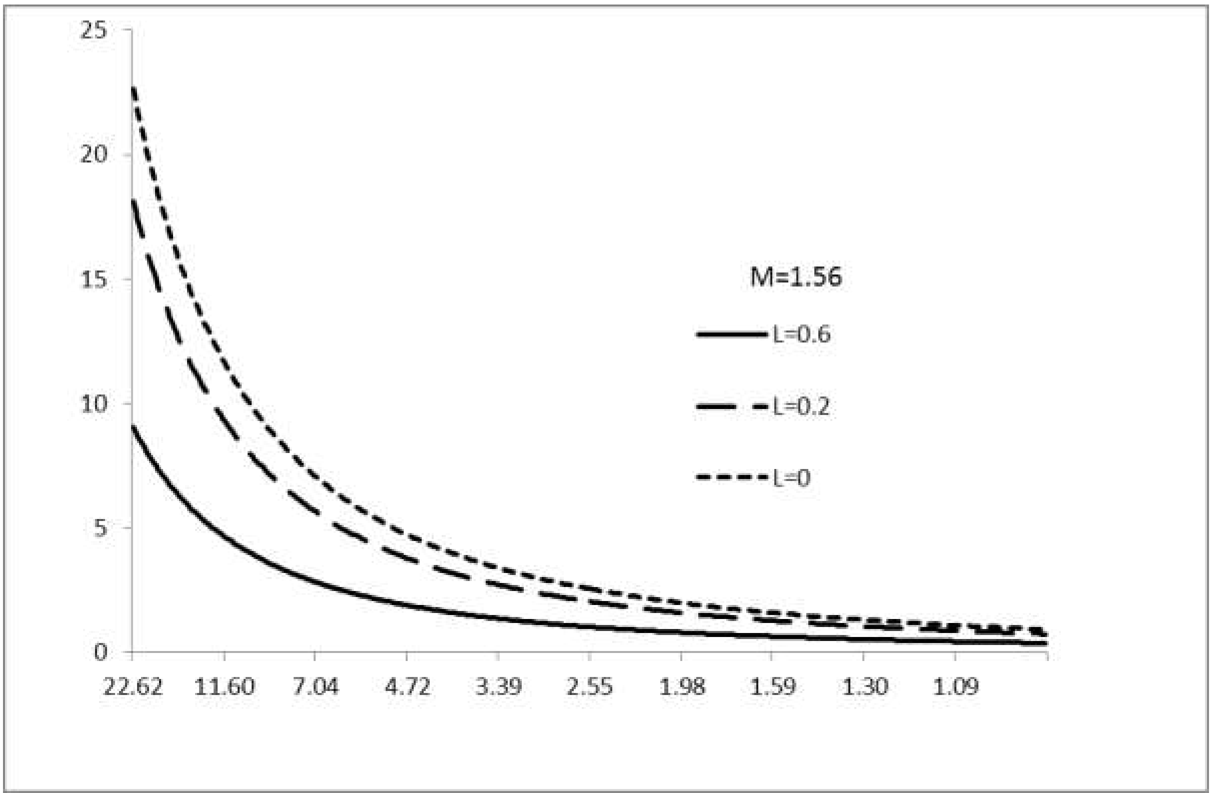

Fig.2 Variation of Displacement $\frac{u}{ka} = U$ with z given kr = 6.5 , $\nu = 0.25$ and I=0.0,0.2,0.6

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